Quadratic Models; Building Quadratic Functions from Data

Example: Maximizing Revenue

In economics, revenue R, in dollars, is defined by the amount of money received from the sale of an item and is equal to the selling price p, in dollars, of the item times the number x of units actually sold. That is, $\boldsymbol{R = xp}$.

The price p in dollars and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{3}x + 100,\ 0 \le x \le 300.$$

a) Express the revenue R as a function of x.
b) What is the revenue if 100 units are sold?
c) What quantity x maximizes revenue? What is the maximum revenue?
d) What price should the company charge to maximize revenue?

a) $R = xp$

$R = x\left(-\frac{1}{3}x + 100\right)$

$\boxed{R(x) = -\frac{1}{3}x^2 + 100x}$

b) $R(100) = -\frac{1}{3}(100)^2 + (100)(100)$

$= \boxed{\$\ 6{,}666.67}$

c) $x = -\frac{b}{2a}$

$x = \frac{-100}{2\left(-\frac{1}{3}\right)} = \frac{-100}{-\frac{2}{3}} = -100 \cdot -\frac{3}{2} = \boxed{150 \text{ units}}$

$R(150) = -\frac{1}{3}(150)^2 + 100(150) = \boxed{\$7500}$

The maximum revenue is \$7500 when 150 units are sold

d) $R = xp$

$7500 = 150p \Rightarrow \boxed{p = \$50}$ or $p = -\frac{1}{3}(150) + 100$

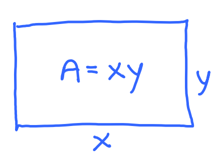
$\boxed{p = \$50}$

The maximum revenue will be reached if the price is \$50.

Example: Maximizing the Area Enclosed by a Fence

Beth has 3000 feet of fencing available to enclose a rectangular field.

a) Express the area A of the rectangle as a function of x, where x is the length of the rectangle.
b) For what value of x is the area largest?
c) What is the maximum area?
d) What are the dimensions of the field with the maximum area?



$P = 2x + 2y = 3000 \text{ ft}$

$\frac{2y}{2} = \frac{3000 - 2x}{2}$

$y = 1500 - x$

a) $A = xy$

$A(x) = x(1500 - x)$

$\boxed{A(x) = -x^2 + 1500x}$

b) $x = \frac{-b}{2a} = \frac{-1500}{2(-1)} = 750$

Area is maximized when $\boxed{x = 750 \text{ ft}}$

c) $A(750) = -(750^2) + 1500(750) = \boxed{562{,}500 \text{ ft}^2}$

d) $y = 1500 - x$

$y = 1500 - 750$

$y = 750 \text{ ft}$

The field with the maximum area is $\boxed{750 \text{ ft} \times 750 \text{ ft}}$

Example: Analyzing the Motion of a Projectile

A projectile is fired at an inclination of 45° to the horizontal, with a muzzle velocity of 100 feet per second. The height h of the projectile is given by $h(x) = \dfrac{-32x^2}{(100)^2} + x,$ where x is the horizontal distance of the projectile from the firing point.

a) At what horizontal distance from the firing point is the height of the projectile a maximum?
b) Find the maximum height of the projectile.
c) At what horizontal distance from the firing point will the projectile strike the ground?
d) Using a graphing calculator, graph the function h, $0 \le x \le 350$.
e) Using a graphing calculator, verify the results obtained in parts b) and c).

$h(x) = \frac{-32}{100^2}x^2 + x$

$h(x) = -.0032x^2 + x$

a) $x = \frac{-1}{2(-.0032)} = \boxed{156.25 \text{ ft}}$

b) $h(156.25) = (-.0032)(156.25)^2 + 156.25 = \boxed{78.125 \text{ ft}}$

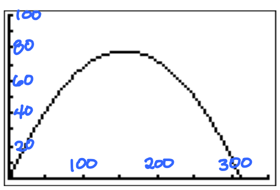
c) When the projectile hits the ground, h=0.

$0 = -.0032x^2 + x$

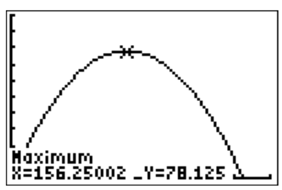
$0 = x(-.0032x + 1)$

$x = 0$ or $x = \frac{1}{.0032} = \boxed{312.5 \text{ ft}}$

d)

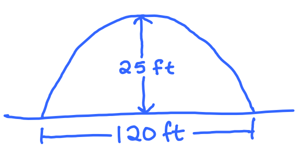


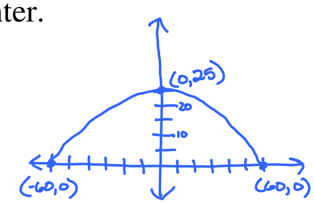
e)



Example: A Parabolic Arch

A parabolic arch has a span of 120 feet and a maximum height of 25 feet. Choose a suitable rectangular coordinate axes and find the equation of the parabola. Then calculate the height of the arch at points 10 feet, 20 feet, and 40 feet from the center.





$f(x) = a(x-h)^2 + k$

Plug in vertex:

$f(x) = ax^2 + 25$

Plug in (60,0) for (x,y):

$0 = a(60)^2 + 25$

$3600a = -25$

$a = \frac{-25}{3600} = -\frac{1}{144}$

$\boxed{f(x) = -\frac{1}{144}x^2 + 25}$

$f(10) = -\frac{1}{144}(10)^2 + 25 = \boxed{24.3 \text{ ft}}$

$f(20) = -\frac{1}{144}(20)^2 + 25 = \boxed{22.2 \text{ ft}}$

$f(40) = -\frac{1}{144}(40)^2 + 25 = \boxed{13.9 \text{ ft}}$